Student seminar notes week 7

Léo Karlen after the talk of Damien Bridel and Loris Schirar

1 Dirichlet Theorem for algebraic number fields

Previously we proved the **Dirichlet Theorem on primes in Arithmetic progression**.

Theorem 1.1. ***(Dirichlet's Theorem on Primes in Arithmetic Progressions)*** *If m is a positive integer and a is an integer for which $(a, m) = 1$, then there are infinitely many primes p satisfying $p \equiv a \,(mod\, m)$.*

The main goal of this section is to generalize this statement to any algebraic number field F, following the proof of Dirichlet, which involved several useful concepts, such as Dirichlet characters and L-functions. Their definition involved the group $\left(\mathbb{Z}/m\mathbb{Z}\right)^{\times}$ which is the Galois group of the cyclotomic extension $\mathbb{Q}(\xi_m)/\mathbb{Q}$. We first need to generalize these notions to our new setting, namely when F is an algebraic number field and $\mathfrak{m}$ is a non-zero integral ideal of $\mathcal{O}_F$.

1.1 Ray class groups

We first focus on the Galois group $\left(\mathbb{Z}/m\mathbb{Z}\right)^{\times}$. We could work with the ideal class group, but it turns out not to be enough. Instead we introduce a new group, named the *Ray class group(s)* which is a generalization of both the class group and the Galois group.

Recall that if an element $\alpha \in F$ satisfies $\sigma(\alpha) > 0$ for every real embedding $\sigma : F \to \mathbb{R}$, we say that α is *totally positive* and write $\alpha \gg 0$. Given a non-zero integral ideal $\mathfrak{m}$ of $\mathcal{O}_F$, we write

$$\mathcal{I}_F(\mathfrak{m}) = \{\mathfrak{a} \mid ord_{\mathfrak{p}}(\mathfrak{a}) = 0 \;\forall\; \mathfrak{p}|\mathfrak{m}\} \subseteq \mathcal{I}_F.$$

On the other hand, we set $\mathcal{P}^{+}_{F,\mathfrak{m}}$ to be the subgroup of $\mathcal{P}_F$ generated by

$$\{\langle\alpha\rangle \mid \alpha \in \mathcal{O}_F,\ \alpha \equiv 1 \,(\mathrm{mod}\,\mathfrak{m}),\ \text{and}\ \ \alpha \gg 0\} \subseteq \mathcal{P}_F.$$

Remark 1.2. We have

$$\begin{aligned}\mathcal{P}^{+}_{F,\mathfrak{m}} &= \{\langle\alpha\rangle \mid \alpha \in F^{\times},\ \alpha \gg 0,\ \alpha \overset{\times}{\equiv} 1 \ (\mathrm{mod}\ \mathfrak{m})\} \\ &= \left\{ \left\langle \frac{\alpha}{\beta} \right\rangle \;\middle|\; \frac{\alpha}{\beta} \gg 0,\ \alpha, \beta \in \mathcal{O}_F \ \text{ prime to } \mathfrak{m},\ \alpha \equiv \beta \ (\mathrm{mod}\ \mathfrak{m}) \right\}.\end{aligned}$$

The proof of these equalities is left as an exercise in exercise sheet 6.

Similarly, we define the *ray modulo* $\mathfrak{m}$ to be

$$\mathcal{P}_{F,\mathfrak{m}} = \{\langle\alpha\rangle \mid \alpha \in F^\times,\ \alpha \overset{\times}{\equiv} 1 \pmod{\mathfrak{m}}\}.$$

The total positivity condition of $\mathcal{P}^+_{F,\mathfrak{m}}$ can be interpreted as a congruence condition for the primes at infinity.

Definition 1.3. (Ray class group) We may finally define the *Ray class group of* F for $\mathfrak{m}$ to be

$$\mathcal{R}_{F,\mathfrak{m}} = {}^{\mathcal{I}_F(\mathfrak{m})}\Big/_{\mathcal{P}_{F,\mathfrak{m}}}.$$

while the *strict (or narrow) ray class group* of F is

$$\mathcal{R}^+_{F,\mathfrak{m}} = {}^{\mathcal{I}_F(\mathfrak{m})}\Big/_{\mathcal{P}^+_{F,\mathfrak{m}}}.$$

The links between those different groups are summed up in the following diagram.

$$\begin{tikzcd}
 & \mathcal{I}_F & \\
\mathcal{P}_F \arrow[ur, hook] & & \mathcal{I}_F(\mathfrak{m}) \arrow[ul, hook'] \\
 & \mathcal{P}_{F,\mathfrak{m}} \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}$$

In the base setting, we recover the usual definition, which confirms that this is indeed a generalization of the Galois group.

Example 1.4. Let $F = \mathbb{Q}$, $\mathfrak{m} = m\mathbb{Z}$. Using (1.2), we have

$$\mathcal{P}^+_{\mathbb{Q},\mathfrak{m}} = \left\{ \left\langle \frac{a}{b} \right\rangle \middle|\, a, b \equiv 1 \pmod{m},\ \frac{a}{b} > 0 \right\}.$$

An isomorphism $\mathcal{P}^+_{\mathbb{Q},\mathfrak{m}} \cong \left({}^{\mathbb{Z}}/_{m\mathbb{Z}}\right)^\times$ is given by the map induced by

$$\begin{array}{ccc} \mathcal{I}_{\mathbb{Q}}(m\mathbb{Z}) & \longrightarrow & \left({}^{\mathbb{Z}}/_{m\mathbb{Z}}\right)^\times \\ \langle \frac{a}{b} \rangle & \longmapsto & ab^{-1} \end{array}.$$

Example 1.5. Let $\mathfrak{m} = \mathcal{O}_F$. One has

$$\begin{aligned} \mathcal{R}_{F,\mathfrak{m}} &= {}^{\mathcal{I}_F}/_{\mathcal{P}_F} = \mathcal{C}_F,\ \text{the ordinary ideal class group} \\ \mathcal{R}^+_{F,\mathfrak{m}} &= {}^{\mathcal{I}_F}/_{\mathcal{P}^+_F} = \mathcal{C}_F,\ \text{the strict (narrow) ideal class group.} \end{aligned}$$

Therefore the ray class groups are indeed generalizations of both the Galois group and the class group.

1.2 Class fields

We now will define the field extension analog to the cyclotomic extension $\mathbb{Q}(\xi_m)/\mathbb{Q}$, named the *class field.*

We first fix some group $\mathcal{P}_{F,\mathfrak{m}}^{+} \leq \mathcal{H} < \mathcal{I}_F(\mathfrak{m})$. Any such group is named an *ideal group mod* $\mathfrak{m}$.

Theorem 1.6. ***(Existence and Definition of Class field)*** *Let $\mathcal{H}$ be an ideal group mod $\mathfrak{m}$. There is a unique Galois extension K/F such that*

$$\begin{aligned}\mathcal{S}_{K/F} &= \{\textit{primes } \mathfrak{p} \textit{ of } \mathcal{O}_F \mid \mathfrak{p} \textit{ splits completely in } K/F\} \\ &\approx \{\textit{primes } \mathfrak{p} \textit{ of } \mathcal{O}_F \mid \mathfrak{p} \in \mathcal{H}\}\,.\end{aligned}$$

We call such an extension the class field of $\mathcal{H}$. (Recall that for two sets $\mathcal{S}, \mathcal{T}$, one writes $\mathcal{S} \approx \mathcal{T}$ if and only if they differ by a set with Dirichlet density zero.)

Proof. The existence part of the theorem is one of the main results of Class Field Theory and will be proven later in the course. For now, we will only prove the uniqueness. Recall that the Dirichlet density of a set $\mathcal{S}$ of primes is

$$\delta_F(\mathcal{S}) = \lim_{s \to 1^+} \frac{\sum_{\mathfrak{p} \in \mathcal{S}} \frac{1}{N\mathfrak{p}^s}}{\log\left(\frac{1}{s-1}\right)}.$$

We moreover know from exercise sheet 6, exercise 2 that $\delta_F(\mathcal{S}_{K/F}) = [K : F]^{-1}$. Let $K_1, K_2/F$ be two field extensions satisfying the theorem's condition. The composite field $K = K_1K_2$ is also a class field. We know from exercise 3 of sheet 6[1] that

$$\mathcal{S}_{K/F} = \mathcal{S}_{K_1/F} \cap \mathcal{S}_{K_2/F} \approx \{\mathfrak{p} \text{ of } F : \ \mathfrak{p} \in \mathcal{H}\}$$

which yields

$$\mathcal{S}_{K/F} \approx \mathcal{S}_{K_1/F} \approx \mathcal{S}_{K_2/F}$$

and thus $[K : F] = [K_1 : F] = [K_2 : F]$ and $K = K_1 = K_2$.

□

Remark 1.7. Notice that the class field is unique up to *strict* equality, not up to isomorphism.

Example 1.8. Let $F = \mathbb{Q}$ and $\mathfrak{m} = m\mathbb{Z}$. We have

$$\begin{aligned}\{p\mathbb{Z} \,|\, p\mathbb{Z} \in \mathcal{P}_{\mathbb{Q},m\mathbb{Z}}^{+}\} &\overset{\text{def}}{=} \{p\mathbb{Z} \mid \ p \cong 1 \pmod{m}, \ p > 0\} \\ &= \{p\mathbb{Z} \,|\, p \text{ splits completely in } \mathbb{Q}(\xi_m)\}\end{aligned}$$

---

[1] The exercise is with $F = \mathbb{Q}$, but a similar argument holds for the general case.

1.3 Weber characters

We now come to the generalization of the Dirichlet character, which we will name *Weber characters.* It is natural to define a Weber character to be a group character $\chi\colon \mathcal{R}^{+}_{F,\mathfrak{m}} \to \mathbb{C}^{\times}$. We also define anagolously an L-function for these generalized Dirichlet characters

$$L_{\mathfrak{m}}(s,\chi) = \sum_{\substack{\mathfrak{a}\in\mathcal{I}_F(\mathfrak{m})\\ \mathfrak{a} \text{ integral}}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}.$$

In the base setting, we used the fact that $\left({}^{\mathbb{Z}}\!/_{m\mathbb{Z}}\right)^{\times}$ is finite to get that any character χ must take value in the unit circle, a useful fact to ensure convergence. As it turns out, a similar argument works in our generalized setting, as $\mathcal{R}^{+}_{F,\mathfrak{m}}$ is also finite.

Theorem 1.9. *$\mathcal{R}^{+}_{F,\mathfrak{m}}$ is a finite group*

Proof. See Childress, Proposition 3.2.1, p.50 or exercise 1 of the exercise sheet of week 7. An explicit formula for the cardinal of $\mathcal{R}^{+}_{F,\mathfrak{m}}$ depending on F and $\mathfrak{m}$ is even provided.

□

Hence $L_{\mathfrak{m}}(s,\chi)$ is well-defined. The following summarizes these new definitions.

Base case	***General case***
$F = \mathbb{Q}$	F algebraic number field
$\mathfrak{m} = m\mathbb{Z}$	$\mathfrak{m}$ non-zero integral ideal of $\mathcal{O}_F$
Galois group $\left({}^{\mathbb{Z}}\!/_{m\mathbb{Z}}\right)^{\times}$	ray class group $\mathcal{R}^{+}_{F,\mathfrak{m}}$
of the cycl. extension $\mathbb{Q}(\xi_m)/\mathbb{Q}$	Class field K/F
Dirichlet characters / L-functions.	Weber characters / L-functions.

We are now able to write the desired generalization of theorem (1.1).

Theorem 1.10. ***(Generalization of Dirichlet's theorem on Primes in Arithmetic Progressions)*** *Given $\mathfrak{a} \in \mathcal{I}_{F,\mathfrak{m}}$, the set*

$$\mathcal{S}_{\mathfrak{a},\mathfrak{m}} = \left\{ \textit{primes } \mathfrak{p} \textit{ of } \mathcal{O}_F \mid \mathfrak{p} \equiv \mathfrak{a} \textit{ in } \mathcal{R}^{+}_{F,\mathfrak{m}} \right\}$$

is infinite.

1.4 Towards the proof of the theorem

We first enumerate some facts about the newly defined L-functions.

(1.) The Euler product formula holds for $\mathrm{Re}(s) > 1$.

$$L_{\mathfrak{m}}(s,\chi) = \sum_{\substack{\mathfrak{a}\in\mathcal{I}_F(\mathfrak{m})\\ \mathfrak{a} \text{ integral}}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}\nmid\mathfrak{m}} \left(1 - \chi(\mathfrak{p})N\mathfrak{p}^{-s}\right)^{-1}.$$

(2.) For $\chi \neq \chi_0$, $L_{\mathfrak{m}}(s,\chi)$ is analytic.

(3.) $L_{\mathfrak{m}}(s,\chi_0)$ has a simple pole at $s = 1$ and is analytic elsewhere.

The proof of these facts are generalizations of the arguments used to prove their equivalent for Dirichlet L-functions.

Proposition 1.11. *Let $\mathfrak{a} \in \mathcal{I}_F(\mathfrak{m})$ and $\mathcal{P}^+_{F,\mathfrak{m}} \leq \mathcal{H} < \mathcal{I}_F(\mathfrak{m})$. If for all Weber characters $\chi \neq \chi_0$ that are trivial on $\mathcal{H}$, we have $L_{\mathfrak{m}}(1,\chi) \neq 0$, then*

$$\delta_F\left(\{\textit{primes } \mathfrak{p} \textit{ of } \mathcal{O}_F \mid \mathfrak{p} \in \mathfrak{a}\mathcal{H}\}\right) = \frac{1}{[\mathcal{I}_F(\mathfrak{m}) :\, \mathcal{H}]}.$$

Proof. Using the Euler product formula for $L_{\mathfrak{m}}(s,\chi)$ and taking the log, we have

$$\log L_{\mathfrak{m}}(s,\chi) = -\sum_{\mathfrak{p}\nmid\mathfrak{m}} \log\left(1 - \chi(\mathfrak{p})N\mathfrak{p}^{-s}\right) = \sum_{\mathfrak{p}\nmid\mathfrak{m}}\sum_{n=1}^{\infty} \frac{\chi(\mathfrak{p})^n N\mathfrak{p}^{-ns}}{n}$$

where we took the Taylor expansion of $\log(1-x)$ for the second equality. Splitting the sum yields

$$= \sum_{\mathfrak{p}\nmid\mathfrak{m}} \chi(\mathfrak{p})N\mathfrak{p}^{-\mathfrak{s}} + \sum_{\mathfrak{p}\nmid\mathfrak{m}}\sum_{n=2}^{\infty} \frac{\chi(\mathfrak{p})^n N\mathfrak{p}^{-ns}}{n} \sim \sum_{\mathfrak{p}\nmid\mathfrak{m}} \chi(\mathfrak{p})N\mathfrak{p}^{-s}.$$

Since χ vanishes on $\mathcal{H}$, it can be seen as a character of $\mathcal{I}_F(\mathfrak{m})\big/\mathcal{H}$. Given a fixed prime $\mathfrak{p}$ of $\mathcal{O}_F$, we have the following orthogonal relation

$$\sum_{\chi} \chi(\mathfrak{a})^{-1}\chi(\mathfrak{p}) = \begin{cases} 0 & \text{if } \mathfrak{p} \notin \mathfrak{a}\mathcal{H} \\ [\mathcal{I}(\mathfrak{m}) :\, \mathcal{H}] & \text{if } \mathfrak{p} \in \mathfrak{a}\mathcal{H} \end{cases}$$

where the sum is taken over characters of $\mathcal{I}_F(\mathfrak{m})\big/\mathcal{H}$. Let

$$\beta_\chi(s) = \sum_{\mathfrak{p}\nmid\mathfrak{m}}\sum_{n=2}^{\infty} \frac{\chi(\mathfrak{p})^n N\mathfrak{p}^{-ns}}{n}.$$

Using the definition and the orthogonal relation, we get

$$\sum_{\chi} \chi(\mathfrak{a})^{-1} \log L_{\mathfrak{m}}(s,\chi) = \sum_{\chi} \chi(\mathfrak{a})^{-1}\left[\sum_{\mathfrak{p}\nmid\mathfrak{m}} \chi(\mathfrak{p})N\mathfrak{p}^{-s} + \beta_\chi(s)\right] \tag{1}$$

$$\sum_{\chi} \chi(\mathfrak{a})^{-1}\left[\log L_{\mathfrak{m}}(s,\chi) - \beta_\chi(s)\right] = \sum_{\mathfrak{p}\in\mathfrak{a}\mathcal{H}} [\mathcal{I}_F(\mathfrak{m}) :\, \mathcal{H}]\, N\mathfrak{p}^{-s}. \tag{2}$$

We write

$$\mathcal{S} = \{\text{primes } \mathfrak{p} \text{ of } \mathcal{O}_F \mid \mathfrak{p} \in \mathfrak{a}\mathcal{H}\}.$$

Our goal is to show that

$$\delta_F(\mathcal{S}) = \lim_{s\to 1^+} \frac{\sum_{\mathfrak{p}\in\mathcal{S}} N\mathfrak{p}^{-s}}{\log\left(\frac{1}{s-1}\right)} = \frac{1}{[\mathcal{I}_F(\mathfrak{m}) : \mathcal{H}]}.$$

Let $h = [\mathcal{I}_F(\mathfrak{m}) : \mathcal{H}]$. By rearranging (2), we get

$$\begin{aligned}\sum_{\mathfrak{p}\in\mathcal{S}} N\mathfrak{p}^{-s} - \frac{1}{h}\log\left(\frac{1}{s-1}\right) =& \frac{1}{h}\sum_{\chi\neq\chi_0} \chi(\mathfrak{a})^{-1}\left[\log L_{\mathfrak{m}}(s,\chi) - \beta_{\xi}(s)\right] \\ &+ \frac{1}{h}\log\left[(s-1)L_{\mathfrak{m}}(s,\chi_0)\right] - \frac{1}{h}\beta_{\chi_0}(s).\end{aligned}$$

Notice that the RHS is bounded as $s \to 1^+$, since by assumption $L_{\mathfrak{m}}(1,\chi) \neq 0$ for every Weber characters different from χ_0 vanishing on $\mathcal{H}$, and we know for a fact that $L_{\mathfrak{m}}(s,\chi_0)$ has a simple pole at $s = 1$. By dividing by $\log\left(\frac{1}{s-1}\right)$ on both sides and using the fact that $\lim_{s\to 1^+} \log\left(\frac{1}{s-1}\right) = \infty$, we get

$$\frac{\sum_{\mathfrak{p}\in\mathcal{S}} N\mathfrak{p}^{-s}}{\log\left(\frac{1}{s-1}\right)} - \frac{1}{h} \to 0 \quad \text{as} \quad s \to 1^+.$$

This amounts to $\delta_F(\mathcal{S}) = \frac{1}{h} = \frac{1}{[\mathcal{I}_F(\mathfrak{m} : \mathcal{H})]}$, as desired.

$\square$

Remark 1.12. In the same setting as in the proposition, if $\mathcal{H} = \mathcal{P}^+_{F,\mathfrak{m}}$, one has

$$\mathcal{S} = \mathcal{S}_{\mathfrak{a},\mathfrak{m}}, \ \ \delta_F(\mathcal{S}) = \frac{1}{|\mathcal{R}^+_{F,\mathfrak{m}}|}.$$

The next step is to show that the assumption are satisfied, namely that $L_{\mathfrak{m}}(1,\chi) \neq 0$ whenever $\chi \neq \chi_0$ vanishes on $\mathcal{H}$. It is nearly accomplished by the next theorem.

Theorem 1.13. *Suppose K/F is a Galois extensions, and $\mathcal{P}^+_{F,\mathfrak{m}} \leq \mathcal{H} < \mathcal{I}_F(\mathfrak{m})$. Suppose there is some set of primes $\mathcal{J} \subseteq \mathcal{H}$, with $\delta_{K/F} \approx \mathcal{J}$. Then*

$$[\mathcal{I}_F(\mathfrak{m}) : \mathcal{H}] \leq [K : F]$$

and $L_{\mathfrak{m}}(1,\chi) \neq 0$ whenever $\chi \neq \chi_0$ and χ is trivial on $\mathcal{H}$.

Proof. Let $m(\chi) = \operatorname{ord}_{s=1}(L_{\mathfrak{m}}(s,\chi))$. We know that $m(\chi) \geq 0$ for $\chi \neq \chi_0$ and $m(\chi_0) = -1$.

Again, since χ is trivial on $\mathcal{H}$, we may view it as a character of $\mathcal{I}_F(\mathfrak{m})\big/\mathcal{H}$. There is some non-zero constant $a \in \mathbb{C}$ such that

$$\prod_{\chi} L_{\mathfrak{m}}(s,\chi) = a(s-1)^{\sum_\chi m(\chi)} + \cdots \sim a(s-1)^{\sum_\chi m(\chi)}$$

where the product is taken over characters of $\mathcal{I}_F(\mathfrak{m})\big/\mathcal{H}$. Taking log of both sides yields

$$\sum_{\chi} \log L_{\mathfrak{m}}(s,\chi) \sim \left(\sum_{\chi} m(\chi)\right)\log(s-1) = -\left(\sum_{\chi} m(\chi)\right)\log\left(\frac{1}{s-1}\right).$$

We have seen in the previous proof that

$$\log L_{\mathfrak{m}}(s,\chi) \sim \sum_{\mathfrak{p}\nmid\mathfrak{m}} \chi(\mathfrak{p})N\mathfrak{p}^{-s}.$$

Hence

$$\sum_{\chi} \log L_{\mathfrak{m}}(s,\chi) \sim \sum_{\mathfrak{p}\in\mathcal{H}} [\mathcal{I}_F(\mathfrak{m}) :\, \mathcal{H}]\, N\mathfrak{p}^{-s}$$

which yields

$$\sum_{\mathfrak{p}\in\mathcal{H}} [\mathcal{I}_F(\mathfrak{m}) :\, \mathcal{H}]\, N\mathfrak{p}^{-s} \sim -\left(\sum_{\chi} m(\chi)\right)\log\left(\frac{1}{s-1}\right). \tag{3}$$

Splitting the sum on the LHS gives us

$$\sum_{\mathfrak{p}\in\mathcal{H}} [\mathcal{I}_F(\mathfrak{m}) :\, \mathcal{H}]\, N\mathfrak{p}^{-s} = [\mathcal{I}_F(\mathfrak{m}) :\, \mathcal{H}]\left(\sum_{\mathfrak{p}\in\mathcal{J}} N\mathfrak{p}^{-s} + \sum_{\mathfrak{p}\in\mathcal{H}\setminus\mathcal{J}} N\mathfrak{p}^{-s}\right)$$

Dividing by $\log\left(\frac{1}{s-1}\right)$ on both sides of (3) and letting $s\to 1^+$, we get

$$-\left(\sum_{\chi} m(\chi)\right) = [\mathcal{I}_F(\mathfrak{m}) :\, \mathcal{H}]\, \delta_F(\mathcal{J}) + [\mathcal{I}_F(\mathfrak{m}) :\, \mathcal{H}] \lim_{s\to 1^+} \frac{\sum_{\mathfrak{p}\in\mathcal{H}\setminus\mathcal{J}} N\mathfrak{p}^{-s}}{\log\left(\frac{1}{s-1}\right)}$$

By assumption, $\mathcal{J} \approx \mathcal{S}_{K/F}$. Moreover, since $\delta_F(\mathcal{S}_{K/F})$ exists and the LHS is finite, the limit on the RHS must exist. Hence one has

$$\begin{aligned}-\left(\sum_{\chi} m(\chi)\right) &= [\mathcal{I}_F(\mathfrak{m}) :\, \mathcal{H}]\, \delta_F(\mathcal{S}_{K/F}) + [\mathcal{I}_F(\mathfrak{m}) :\, \mathcal{H}]\,(\text{some finite nonnegative constant})\\ &\geq [\mathcal{I}_F(\mathfrak{m}) :\, \mathcal{H}]\, \delta_F(\mathcal{S}_{K/F}) = \frac{[\mathcal{I}_F(\mathfrak{m}) :\, \mathcal{H}]}{[K :\, F]}.\end{aligned}$$

Since $m(\chi)\geq 0$ and $m(\chi_0) = -1$, we get

$$1 \geq 1 - \sum_{\chi\neq\chi_0} m(\chi) \geq \frac{[\mathcal{I}_F(\mathfrak{m}) :\, \mathcal{H}]}{[K :\, F]} > 0. \tag{4}$$

In particular

$$0 \geq -\sum_{\chi \neq \chi_0} m(\chi) > -1$$

which forces $m(\chi) = 0$ for all $\chi \neq \chi_0$. Hence $-\left(\sum_\chi m(\chi)\right) = 1$ and $L_{\mathfrak{m}}(1, \chi)$ has a non-zero constant term when expanded in powers of $(s-1)$. In particular $L_{\mathfrak{m}}(1, \chi) \neq 0$ for every $\chi \neq \chi_0$. On the other hand, one has from (4) that

$$1 \geq \frac{[\mathcal{I}_F(\mathfrak{m}) : \mathcal{H}]}{[K : F]}$$

which gives $[K : F] \geq [\mathcal{I}_F(\mathfrak{m}) : \mathcal{H}]$, as desired.

□

Remark 1.14. If we assume the class field of $\mathcal{H}$ over F to exist and let

$$\mathcal{J} = \{\text{primes } \mathfrak{p} \text{ in } \mathcal{O}_F \text{ such that } \mathfrak{p} \in \mathcal{H}\} \subseteq \mathcal{H},$$

one has $\mathcal{J} \approx \mathcal{S}_{K/F}$ and we may prove the generalization of the Theorem on Primes in Arithmetic Progressions (Theorem 1.10) using this last result.